8.5 **Diffraction by a circular aperture**

Plane wave impinges on a screen Σ containing a circular aperture and the consequent far field diffraction pattern spreads across a distant observing screen σ . Recall that the total electric field in point P(X,Y,Z) is calculated by

$$E = \frac{\varepsilon_A \, e^{i(\omega t - kR)}}{R} \iint_{aperture} e^{ik(Yy+Zz)/R} dS \qquad (8.5)$$

For a circular opening, symmetry would suggest introducing polar coordinates in both the plane of the aperture and the plane of the observation, as shown in fig 12. Let

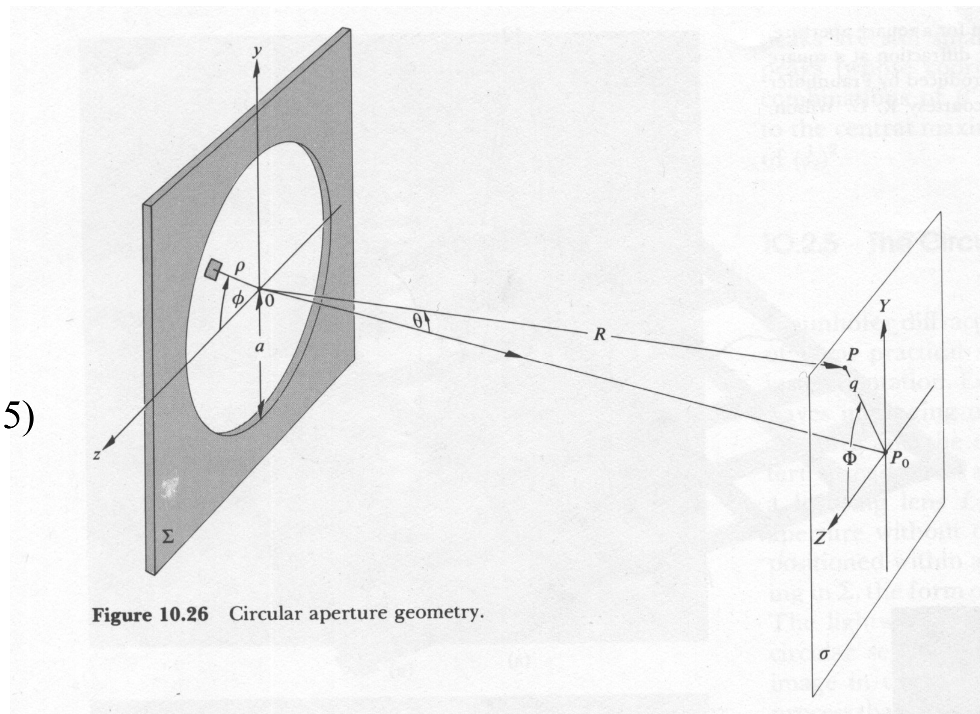

Fig 12 Circular aperture geometry

$$z = \rho\cos\phi \qquad y = \rho\sin\phi$$

$$Z = q\cos\Phi \qquad Y = q\sin\Phi$$

$$dS = \rho d\rho d\phi$$

Substituti ng these e xpressions into Eq. (8.5), it becomes

$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R}\int_{\rho=0}^{a}\int_{\varphi=0}^{2\pi} e^{i(k\rho q/R)\cos(\varphi-\Phi)}\rho d\rho d\varphi \qquad (8.21)$$

Because of the complete axial symmetry, the solution must be independent of Φ. We can solve Eq. (8.21) with$\Phi = 0$. The portion of the double integral associated with the variable ϕ is

$$\int_0^{2\pi} e^{i(k\rho q/R)\cos\phi}d\phi$$

This is 0^{th} order Bessel function of the first kind, the general form of order m is defined as

$$J_m(u) = \frac{i^{-m}}{2\pi}\int_0^{2\pi} e^{i(mv+u\cos v)}dv \qquad (8.22)$$

Equation (8.21) cab be rewritten as

$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R}2\pi\int_0^a J_0(k\rho q/R)\rho d\rho \qquad (8.23)$$

Bessel functions have the following recurrence property

$$\frac{d}{du}[u^m J_m(u)] = u^m J_{m-1}(u) \qquad (8.24)$$

When m=1, this leads to

$$\int_0^u u' J_0(u')du' = uJ_1(u) \qquad (8.25)$$

Let $w = k\rho q / R$, Eq. (8.23) becomes

$$\int_{\rho=0}^{\rho=\mathrm{a}} J_0(k\rho q / R)\rho d\rho = \int_{w=0}^{w=kaq/R} J_0(w)wdw$$

Making use of Eq. (8.25), we get

$$\mathrm{E} = \frac{\varepsilon_{\mathrm{A}} e^{i(\omega t - kR)}}{R} 2\pi a^2 (R / kaq) J_1(kaq / R) \qquad (8.26)$$

The irradiance at point P is $EE^* / 2$

$$I = 2(\frac{\varepsilon_A A}{R})^2 [\frac{J_1(kaq / R)}{kaq / R}]^2 \qquad (8.27)$$

Where A is the area of the circular opening.

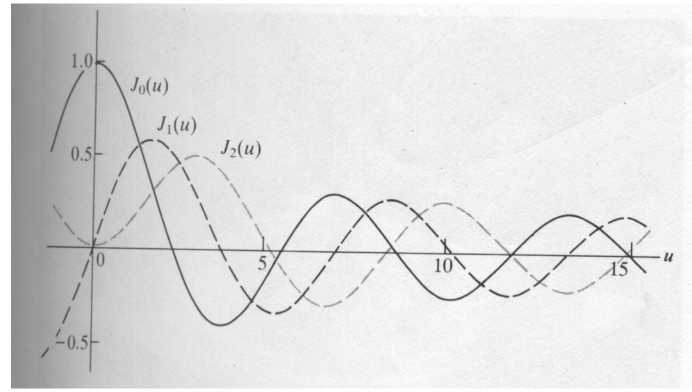

Fig 13 Bessel functions

Bessel functions are shown in figure 13. At the center of the pattern, q=0, so u=kaq/R=0 and $J_1(u)=0$. So

$$\left.\frac{J_1(u)}{u}\right|_{u=0} = \left.\frac{dJ_1(u)/du}{du/du}\right|_{u=0} = \left.\frac{d}{du}J_1(u)\right|_{u=0} \qquad (8.28)$$

From Eq.(8.24) (m=1), $J_0(u) = \frac{d}{du}J_1(u) + \frac{J_1(u)}{u}$, at u=0

$$J_0(u)\big|_{u=0} = 2\left.\frac{J_1(u)}{u}\right|_{u=0}, \text{ or}$$

$$\left.\frac{J_1(u)}{u}\right|_{u=0} = \frac{1}{2}J_0(u)\Big|_{u=0} = 1/2 \qquad (8.29)$$

$$I(0) = \frac{1}{2}\left(\frac{\varepsilon_A A}{R}\right)^2 \qquad (8.30)$$

and since $q/R = \sin\theta$, the irradiance becomes

$$I(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \qquad (8.31)$$

Eq. (8.31) is depicted in figure 8.14. The first zero irradiance corresponds to the first zero of $J_1(u)$ when $u = \frac{2\pi}{\lambda}a\sin\theta_1 = 3.83$, or (D is the aperture diameter)

$$\sin\theta_1 = \frac{3.83\lambda}{\pi 2a} = 1.22\frac{\lambda}{D} \qquad (8.32)$$

When the aperture is a lens with a focal length of f, the radius of the central bright ring will be $q_1 = 1.22\frac{f\lambda}{D}$. There are number of secondary peaks and zeros as shown in figure 14. The central spot contains 84% of the total irradiance.

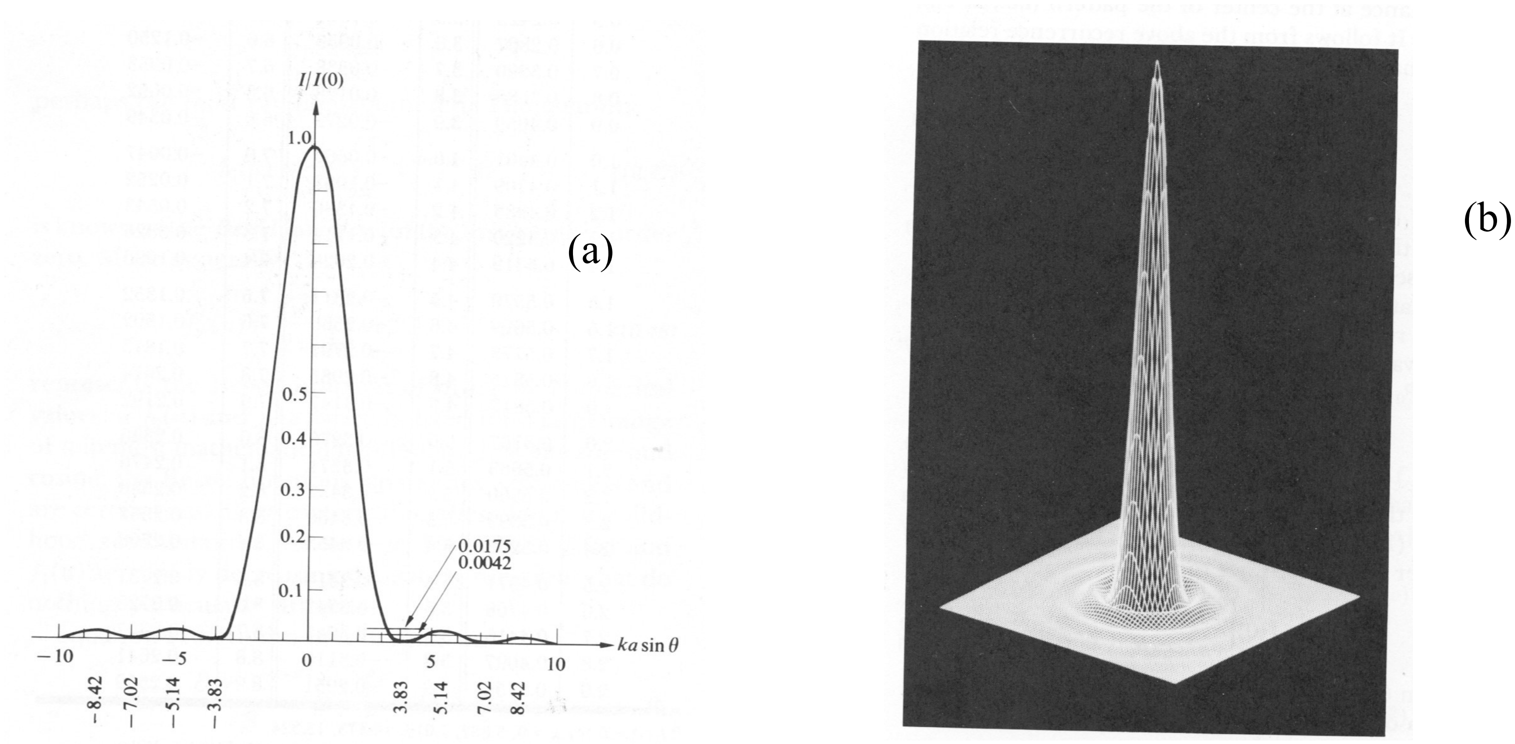

Figure 14 (a) I/I(0) of Eq. (8.31). (b) Irradiance resulting from Fraunhofer diffraction of a circular aperture